Wick's theorem at nonzero temperatures

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The basic formulas for equilibrium thermofield dynamics are derived using an elementary group-theoretical approach. The formalism is then used to derive Wick's theorem as an operator relation at finite temperature.

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I. INTRODUCTION

Wick's theorem, relating time-ordered operators to their normal-ordered form, is a basic theorem in the derivation of diagrammatic perturbation expansion in many-body theory [14]. The usual derivation is given via an operator relation in the zero-temperature, $T=0$ case. In the $T\neq 0$ case the relation is given in terms of expectation values. In the following we show that thermofield dynamics (TFD) which reduces $T\neq 0$ problems to $T=0$ ones (through doubling the number of degrees of freedom) allows the derivation of a $T\neq 0$ operator relation.

As this paper is mainly pedagogical, we present in Sec. II a brief discussion of a peculiarity of quantum mechanics—namely, entangled states (ES). We give the relevance of ES to Bell's inequalities, which are also touched upon in this section. Then the ES's relevance to mixed states is presented. Section III contains a derivation of some results of equilibrium thermofield dynamics in, perhaps, a somewhat novel way. Section IV includes the general perturbation expression for the grand canonical partition function (GCPF) which provides the settings for our main problem, viz., derivation of Wick's theorem at $T\neq 0$ as an operator relation. This is done in Sec. V. Section VI contains some concluding remarks and comments on Bose-Einstein condensation (BEC), as viewed from our approach. Some of the detailed mathematical proofs are relegated to the appendixes: Appendix A contains a summary of (known) results of the SU(1,1) Lie group, while Appendix B provides the detailed proofs of the two steps needed in the derivation of Wick's theorem.

II. ENTANGLED STATES

In this section we define entangled states [1] and give a brief discussion of their relevance to Bell's inequality [2] (BIQ) and to generation of mixed states (MS). The first (BIQ) aims at underscoring the intrinsic quantum-mechanical (QM cal) nature of ES, while the second (MS) is relevant to our problem of nonzero temperature ($T\neq 0$).

An entangled state [1] is a state of two (or more) systems (e.g., particles) which cannot be written as a product. A famous example [3] is the $S=0$, two spin-$\frac{1}{2}$ particles, labeled 1 and 2:

$$|\psi(1,2)\rangle=\frac{1}{\sqrt{2}}[|\uparrow 1\rangle|\downarrow 2\rangle-|\downarrow 1\rangle|\uparrow 2\rangle] . \tag{1}$$

Here the spin space σ of each particle is two dimensional: $(\uparrow,\downarrow)$. That ES, which are so common in QM, are epistemologically difficult was noted by Schrödinger himself, who coined their name.

It is, perhaps, not surprising that such states cannot exist within classical physics. To see that this indeed is the case, we shall now present a brief discussion of Bell's inequality—the violation thereof is a peculiarity of quantum mechanics. Bell [2] noted that for physical systems to possess local objective properties—i.e., such that are independent of observations—implies certain inequalities, the BIQ. Classical physics cannot lead to Bell's inequality violation (BIQV). QM, however, for certain states [e.g., Eq. (1)] may lead to BIQV. Thus, BIQV's can be taken as fingerprints of QM.

Now it has been shown [4,5] that whenever a system is in ES, BIQV can be concocted. Furthermore, if the state is a product state, no BIQV is possible [5]. Thus, we conclude that the presence within QM of ES is the root of this (the BIQV implications) conceptual difficulty. A more general way to prescribe a state in QM is via a density matrix ρ. It was shown by von Neumann [6] long ago (1930s) that a pure QM cal state (i.e., such that $\rho^2=\rho$) of two particles (or systems) can be a mixed state (i.e., $\rho^2\neq\rho$) for one of the constituent particles. Thus for the state $|\psi(1,2)\rangle$ given in Eq. (1) we have

$$\rho(1,2)=|\psi(1,2)\rangle\langle\psi(1,2)| , \tag{2}$$

$$[\rho(1,2)]^2=\rho(1,2) . \tag{3}$$

If we now consider

$$\rho(1)\equiv\sum_{\sigma^2}\langle\sigma^2|\psi(1,2)\rangle\langle\psi(1,2)|\sigma^2\rangle \tag{4}$$

$$=\sum_{\sigma_1}\tfrac{1}{2}|\sigma^1\rangle\langle\sigma^1| , \tag{5}$$

i.e., particle 1, $\rho(1)$, whose density matrix is obtained via tracing out particle-2 coordinates is in a mixed state: $\rho^2(1)\neq\rho(1)$. This can happen only in QM, as can be seen as follows: Defining [6,7] entropy,

$$S=-\mathrm{Tr}(\rho\ln\rho) . \tag{6}$$

We have in QM the Araki-Lieb inequality [7–9]

$$|S(1)-S(2)|\leq S(1,2)\leq S(1)+S(2) . \tag{7}$$

Here,

$$\rho(j)=\mathrm{Tr}_i\rho(i,j),\quad i,j=1 \text{ or } 2 , \tag{8}$$

$$S(j)=-\mathrm{Tr}[\rho(j)\ln\rho(j)] , \tag{9}$$

$$S(i,j)=-\mathrm{Tr}[\rho(i,j)\ln\rho(i,j)] . \tag{10}$$

The "corresponding" classical theorem [7,9] is

$$\max\{S(1),S(2)\}\leq S(1,2)\leq S(1)+S(2) . \tag{11}$$

Thus, whereas classically whenever the entropy of the combined system is zero,

$$S(1,2)=0 . \tag{12}$$

Equation (11) implies $S(1)=S(2)=0$ ($S\geq 0$). In QM the situation is quite different: all that (12) implies is that

$$S(1)=S(2) . \tag{13}$$

Now, obviously, if $\rho(1,2)=\rho\otimes\rho(2)$, then $S(1,2)=S(1)+S(2)$, and $S(1,2)=0$ implies, as in the classical case, $S(1)=S(2)=0$. However, if we deal with an ES, $\rho(1,2)\neq\rho(1)\otimes\rho(2)$. In this case we have that each of the constituent systems is in a mixed state while the combined system is in a pure state.

It is trivial to show that every mixed state can be looked upon as the result of tracing out of the coordinates of another (image) system where both systems (physical and image) were in an appropriately chosen pure state. We argued that the original pure state must be an ES. Thus, for example, a judicial choice of an ES could lead to a thermal density matrix for a physical state.

III. EQUILIBRIUM THERMOFIELD DYNAMICS (REFS. [10,11])

In this section we consider a pure boson ("physical") field in thermal equilibrium. We consider one mode only for simplicity. The Hamiltonian is ($\hbar=1$)

$$H_0=\omega a^\dagger a , \tag{14}$$

with

$$[a,a^\dagger]=1,\quad [a,a]=[a^\dagger,a^\dagger]=0 . \tag{15}$$

H_0 is diagonal in the occupation number representation: $|n\rangle$, $n=0,1,2,\ldots$; $a|0\rangle=0$.

We now introduce an auxiliary ("tilde") field which forms a pure (entangled) state with the physical field and leads, upon being traced out, to a thermal state for the physical field. Standard TFD considerations [10,11] as well as the Araki-Lieb formula [Eq. (7)] suggest for the tilde field the image field. It is described by $\tilde{a}$, the annihilation, and $\tilde{a}^\dagger$, the creation operator, with

$$[\tilde{a},\tilde{a}^\dagger]=1,\quad [\tilde{a},\tilde{a}]=[\tilde{a}^\dagger,\tilde{a}^\dagger]=0 . \tag{16}$$

The occupation space of the tilde field is denoted by $|\tilde{n}\rangle$, $\tilde{n}=0,1,\ldots$; $\tilde{a}|0\rangle=0$. All tilde operators commute with nontilde ones. With the aid of this tilde field, we may write

$$\langle\langle G(a^\dagger,a)\rangle\rangle\equiv\mathrm{Tr}[e^{-\beta H_0}G(a^\dagger,a)]/\mathrm{Tr}e^{-\beta H_0} , \tag{17}$$

with $G(a^\dagger,a)$ an arbitrary function of the (physical) field operators $a,a^\dagger$, as

$$\langle\langle G(a^\dagger,a)\rangle\rangle=\langle\psi_N|G(a^\dagger,a)|\psi_N\rangle . \tag{18}$$

Using [12]

$$|\psi_N\rangle=\frac{e^{-\beta\omega K_z}}{\sqrt{\bar{Z}_N}}\sum_{\substack{m=0\\ \tilde{m}=m+N}}^{\infty}|m,\tilde{m}\rangle , \tag{19}$$

i.e., the thermal expectation value is expressed as an expectation value of $G(a^\dagger,a)$ in a pure entangled state. $N\geq 0$ is an arbitrary integer. Notice that $(\tilde{m}-m)$ may be taken as a constant N, since the Casimir operator $\hat{C}$ of su(1,1) depends only on $(\tilde{m}-m)$. As $\hat{C}$ commutes with all operators in the algebra, it is a constant. In the following and in Appendix A we justify the relevance of su(1,1) of our problem:

$$\bar{Z}_N=\sum_{\substack{m=0\\ \tilde{m}=m+N}}^{\infty}\langle m,\tilde{m}|e^{-\beta\omega K_z}|m,\tilde{m}\rangle , \tag{20}$$

$$K_Z=\tfrac{1}{2}=[a^\dagger a+\tilde{a}\tilde{a}^\dagger] . \tag{21}$$

The validity of Eq. (19) is shown in Appendix A where, in addition, an alternative rationale for expression (19) is given.

A particularly simple expression obtains for $N=0$ [12,13]:

$$|\psi(\beta)\rangle=\frac{1}{\sqrt{\bar{Z}}}e^{-\beta(\omega/2)K_z}e^{K_+}|0,\tilde{0}\rangle ,$$

$$K_+=a^\dagger\tilde{a}^\dagger . \tag{22}$$

Thus,

$$|\psi(\beta)\rangle=\frac{1}{\sqrt{\bar{Z}}}e^{\tanh\gamma K_+}|0,\tilde{0}\rangle , \tag{23}$$

$$\tanh^2\gamma=e^{-\beta\omega} . \tag{24}$$

We note that [13]

$$K_-\equiv(K_+)^\dagger=\tilde{a}a \tag{25}$$

closes an algebra [su(1,1)] with K_z and K_+:

$$[K_\pm,K_z]=\pm K_\pm , \tag{26}$$

$$[K_+,K_-]=-2K_z . \tag{27}$$

Hence we may use this algebra to express

$$U(\gamma)\equiv e^{\gamma K_+-\gamma K_-} \tag{28}$$

in its normal order (i.e., creation operators to the left of annihilation operators):

$$U(\gamma)=e^{\tanh\gamma K_+}e^{-2\ln(ch\gamma)K_Z}e^{-\tanh\gamma K_-} . \quad (29)$$

Hence, direct application on the vacuum ($|0,\tilde{0}\rangle$) state yields

$$|\gamma\rangle \equiv U(\gamma)|0,\tilde{0}\rangle = |\psi(\beta)\rangle . \quad (30)$$

Hence, we have finally

$$\langle\langle G(a^\dagger,a)\rangle\rangle = \langle 0,0|U^\dagger(\gamma)G(a^\dagger,a)U(\gamma)|0,0\rangle . \quad (31)$$

Thus, thermal averaging is equivalent to the zero temperature (i.e., vacuum) expection value of the transformed operator

$$\bar{G}(a^\dagger,a)=U^\dagger(\gamma)G(a^\dagger,a)U(\gamma) . \quad (32)$$

This is the well-known TFD result [10] for the equilibrium case of free bosons.

IV. THE GRAND CANONICAL PARTITION FUNCTION FOR INTERACTING BOSONS

The GCPF for interacting bosons is given by [14]

$$Z(\beta,\mu,V)=\mathrm{Tr}e^{-\beta(H_0+V)} . \quad (33)$$

Here,

$$H_0=\sum_k(\epsilon_k-\mu)a_k^\dagger a_k , \quad (34)$$

$$V=\sum_{g,k,k'}V(q)a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'}a_k . \quad (35)$$

Standard many-body theory procedure [14] allows us to write

$$Z(\beta,\mu,V)=Z_0(\beta,\mu,V)+\left\langle\!\!\left\langle T_\tau\left[\exp\left[-\int_0^\beta V(\tau)dr\right]\right]\right\rangle\!\!\right\rangle_0 . \quad (36)$$

$Z_0(\beta,\mu,V)$ is the GCPF for free particles, i.e., with H_0 as the Hamiltonian, T_τ is the τ ordering operator [14], and

$$V(\tau)=e^{+H_0\tau}Ve^{-H_0\tau} , \quad (37)$$

i.e., in the interaction representation. The double brackets are defined via (A is an arbitrary operator)

$$\langle\langle A\rangle\rangle_0=\mathrm{Tr}(e^{-\beta H_0}A)/\mathrm{Tr}e^{-\beta H_0} . \quad (38)$$

Our argument of Sec. III is now applicable and allows us to equate [henceforth, $|0\rangle$ designates the double vacuum of (30)]

$$\langle\langle A\rangle\rangle=\langle 0|[U(\bar{\gamma})]^\dagger AU(\bar{\gamma})|0\rangle , \quad (39)$$

$$U(\bar{\gamma})=\prod_k U(\gamma_k) , \quad (40a)$$

$$U(\gamma_k)=e^{\gamma_k K_{k+}-\gamma_k K_{k-}} , \quad (40b)$$

$$\tanh^2\gamma_k=e^{-\beta(\epsilon_k-\mu)} , \quad (41)$$

$$K_{k_+}=a_k^\dagger\tilde{a}_k^\dagger,\quad K_{k_-}=(K_{k_+})^\dagger . \quad (42)$$

Thus we have for

$$\left\langle\!\!\left\langle T_\tau\left[\exp\left[-\int_0^\beta V(\tau)dr\right]\right]\right\rangle\!\!\right\rangle_0 = \left\langle\!\!\left\langle 0\left|T_\tau\left[\exp\left[-\int_0^\beta \bar{V}(\tau)dr\right]\right]\right|0\right\rangle\!\!\right\rangle , \quad (43)$$

with

$$\bar{V}(\tau)=U^\dagger(\bar{\gamma})V(\tau)=U(\bar{\gamma}) . \quad (44)$$

Or, defining b operators by

$$b_k=U^\dagger(\gamma_k)a_k(\tau)U(\gamma_k) = [\cosh(\gamma_k)a_k-\sinh(\gamma_k)\tilde{a}^\dagger]e^{-(\epsilon_k-\mu)\tau} , \quad (45)$$

with similar definitions for $b_k^\dagger$, $\tilde{b}_k$, and $\tilde{b}_k^\dagger$, we rewrite the interaction with b's replacing a's. This is possible because the τ dependence of the a's (and hence of the b's) is trivial, e.g.,

$$a_k(\tau)=e^{-(\epsilon_k-\mu)\tau}a_k . \quad (46)$$

The result of the above is that, upon expansion of the exponential in (45) after extracting the τ factors, we must evaluate

$$\langle 0|AB\ldots XYZ|0\rangle , \quad (47)$$

where $A,B,C,\ldots$ are b or $b^\dagger$ and we may consider each mode separately. (The operators appear, of course, in a prescribed order that was dictated by their τ dependence.) We now come to our main point: obtaining an operator relation between $ABC\cdots$ and appropriately defined N operator [14]. To this end, we *define* N ordering of the $b,b^\dagger$ operators as the usual N ordering of their $a,a^\dagger$ constituents, e.g., ($c=\cosh\gamma$, $s=\sinh\gamma$)

$$N(bb^\dagger)=c^2a^\dagger a+s^2\tilde{a}^\dagger\tilde{a}-cs(\tilde{a}^\dagger a^\dagger+a\tilde{a}) \neq b^\dagger b . \quad (48)$$

The reason is that the vacuum is defined as the vacuum of the $a,\tilde{a}$ operators and the above definition assures that

$$\langle 0|N(AB\cdots XYZ)|0\rangle=0,\quad A,B,\ldots:\ b \text{ or } b^\dagger . \quad (49)$$

The contraction of A and B, $\overline{AB}$, is defined as usual:

$$\overline{AB}=AB-N(AB) . \quad (50)$$

Note that the order of AB in the contracted pair is important (as in the usual case [15])

$$\overline{BA}\neq\overline{AB} . \quad (51)$$

We show in Appendix B that $\overline{AB}$ is a C number:

$$\overline{b_{k_1}b_{k_2}^\dagger}=\cosh^2(\gamma_k)\delta_{k_1,k_2} , \quad (52)$$

$$\overline{b_{k_1}^\dagger b_{k_2}}=\sinh^2(\gamma_{k_1})\delta_{k_1,k_2} . \quad (53)$$

All the other contractions are zero. We also show in Appendix B that

$$ABC\cdots XYZ=N(ABC\cdots XYZ)+\overline{AB}N(CD\cdots XYZ) +\overline{AC}N(BD\cdots XYZ)+\cdots +\overline{AB}\,\overline{CD}N(\cdots) +\overline{AB}\,\overline{CD}\cdots\overline{YZ}+\cdots , \quad (54)$$

i.e., a given string of ordered operators is equal to the contractions and normal-ordered operators in an operator formula which is identical in appearance to the one valid at $T=0$. Now (49) implies that the only contributions to (43) are the terms with all pairs contracted. This formula differs from the zero-temperature case only in that whenever $\overline{AB}\neq 0$ for $\tau_A>\tau_B$, it is also nonzero for $\tau_B>\tau_A$, where τ_A is the τ of the operator A, etc. (The case $\tau_A=\tau_B$ is identical to the $T=0$ case, viz., it can only occur for a's already normally ordered.)

Thus we have retrieved the standard diagrammatic expansion via an operator-relation Wick's theorem. Note that $N(ABC\cdots)$ cannot be written as a product of ABC in any order. It is, rather, a function of $A,B,C,\ldots$. We illustrate the above in getting the lowest-order term in perturbation:

$$\sim\langle 0|b_{k_1}^{\dagger}a_{k_2}^{\dagger}b_{k_3}b_{k_4}|0\rangle_{k_1+k_2=k_3+k_4}$$
$$=(\overline{b_{k_1}^{\dagger}b_{k_3}}\;\overline{b_{k_2}^{\dagger}b_{k_4}}+\overline{b_{k_1}^{\dagger}b_{k_4}}\;\overline{b_{k_2}^{\dagger}b_{k_3}})\delta_{k_1+k_2,k_3+k_4}$$
$$=\sinh^2\gamma_{k_1}\sinh^2(\gamma_{k_2})(\delta_{k_1+k_3}\delta_{k_2+k_4}+\delta_{k_1+k_4}\delta_{k_2+k_3})\,. \qquad (55)$$

Thus we recover the standard expression with [cf. Eq. (41)]

$$\sinh^2\gamma_k=\frac{1}{e^{\beta(\epsilon_k-\mu)}-1}\,. \qquad (56)$$

As Eq. (46) implies, the explicit τ dependence is to be added and is the τ dependence of the a's, i.e. [cf. Eq. (45)],

$$b_k(\tau)=e^{-(\epsilon_k-\mu)\tau}[\cosh(\gamma_k)a_k+\sinh(\gamma_k)\tilde a_k^{\dagger}]\,, \qquad (57)$$

$$b_k^{\dagger}(\tau)=e^{+(\epsilon_k-\mu)\tau}[\cosh(\gamma_k)a_k^{\dagger}+\sinh(\gamma_k)\tilde a_k]\,. \qquad (58)$$

This τ dependence assures us the correct τ dependence for the free Green's function:

$$G_k^0(\tau)=-\langle T[b_k(\tau)b_k^{\dagger}(0)]\rangle \qquad (59)$$

$$=\begin{cases}-\overline{b_k(\tau)b_k^{\dagger}(0)}, & \tau>0 \qquad (60)\\ -\overline{b_k^{\dagger}(0)b_k(\tau)}, & \tau<0\,. \qquad (61)\end{cases}$$

These lead to

$$G_k^0(\tau)=\begin{cases}-e^{-(\epsilon_k-\mu)\tau}\cosh^2\gamma_k, & \tau>0 \qquad (62)\\ -e^{-(\epsilon_k-\mu)\tau}\sinh^2\gamma_k, & \tau<0\,. \qquad (63)\end{cases}$$

Thus we have

$$G_k^0(\tau)=G_k^0(\tau+\beta),\quad \tau<0\,. \qquad (64)$$

V. SUMMARY

The relevance of the peculiarly quantum-mechanical states—the entangled states—to the expression of a physical mixed state as a pure state of an image (designated as tilde) system entangled with the physical system was presented. The particular case of a thermal (physical) state was assigned to an explicit entangled state possessing some simple features. This assignment which was given for the case of free (i.e., noninteracting) bosons was used to produce a new derivation of Wick's theorem at finite temperature ($T\neq 0$). This new derivation gives the theorem at $T\neq 0$ as an operator relation in analogy to the case of $T=0$. Of particular interest in this regard is the study (now in progress) of Bose-Einstein condensation as a $T=0$ problem [12].

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APPENDIX A: ENTANGLED STATE FOR A THERMAL STATE

In this Appendix we prove Eq. (19). Consider [cf. Eq. (20)]

$$|\psi_N\rangle=\sum_{\substack{m=0\\ \tilde m=m+N}}^{\infty} e^{-\beta\omega/4}e^{-\beta\omega m/2}e^{(-\beta\omega/2)\tilde m}|m,\tilde m\rangle/\sqrt{\bar Z_N}\,. \qquad (A1)$$

This state is normalized,

$$\langle\psi_N|\psi_N\rangle=1\,. \qquad (A2)$$

Next we calculate the mean value of $G(a^{\dagger},a)$, an arbitrary function of $a^{\dagger},a$:

$$\langle\psi_N|G(a^{\dagger},a)|\psi_N\rangle$$
$$=\frac{1}{\bar Z_N}\sum_{n=0}^{\infty}e^{-\beta\omega/2}e^{(-\beta\omega/2)N}e^{-\beta\omega n}\langle n|G(a^{\dagger},a)|n\rangle$$
$$=\mathrm{Tr}[e^{-\beta\omega a^{\dagger}a}G(a^{\dagger},a)]/\mathrm{Tr}e^{-\beta\omega a^{\dagger}a}\,. \qquad (A3)$$

The reason for choosing the vector space $\{|m,m+N\rangle, m=0,1,\ldots,\infty;\ N\geq 0\text{—a fixed number}\}$ is that this space is invariant under the set of operators

$$K_z=(a^{\dagger}a+\tilde a\tilde a^{\dagger})1/2\,,$$

$$K_+=a^{\dagger}\tilde a^{\dagger},\quad K_-=\tilde a a=(K_+)^{\dagger}\,.$$

The su(1,1) algebra of these operators governs the study of interest here.

APPENDIX B: PROOF OF SOME RELATION

In this appendix we show that (a) the contraction defined by Eq. (5) and our N ordering [see Eq. (48) above] is a c number and (b) Eq. (54) holds.

(a) The contraction. We have $(k \neq k')$

$$\overline{b_k b_{k'}^\dagger} = \overline{b_{k'}^\dagger b_k} = 0 \tag{B1}$$

because the constituent operators commute. For $k = k'$ we have

$$\overline{b_k b_k} = \overline{b_k^\dagger b_k^\dagger} = 0 \tag{B2}$$

because the tilde operators commute with the nontilde ones.

We now consider the cases of $k = k'$ (hence we delete the index k). Note that the τ dependence [Eq. (45)] is irrelevant, and hence is deleted too. We have [cf. Eq. (48)]

$$\begin{aligned} \overline{bb^\dagger} &= (ca - s\tilde{a}^\dagger)(ca^\dagger - s\tilde{a}) - c^2 R^+ a \\ &\quad - s^2 \tilde{a}^\dagger \tilde{a} + cs(\tilde{a}^\dagger a^\dagger + a\tilde{a}) \\ &= \cosh^2\gamma , \end{aligned} \tag{B3}$$

while in a similar way we get

$$\overline{b^\dagger b} = \sinh^2\gamma . \tag{B4}$$

Thus we showed that all the contractions give c numbers.

(b) Proof of Eq. (54). The theorem is trivially true if the indices (k) are distinct, e.g., for $k \neq k_i$, $i = 1, \ldots, N$,

$$b_k N(b_{k_1} b_{k_2}^\dagger \cdots b_{k_N}) = N(b_k b_{k_1} b_{k_2}^\dagger \cdots b_{k_N}) \tag{B5}$$

because all the constituent operators commute; hence, $\overline{b_k b_{k_i}^\dagger} = 0$, and the equation holds. Thus we consider the case of

$$AN(BC \cdots XYZ), \quad \text{with } A,B,C,\ldots = b, b^\dagger , \tag{B6}$$

all referring to the same mode (i.e., of equal index k). We prove Eq. (54) by induction. Thus we assume that

$$\begin{aligned} AN(BC \cdots XY) &= N(ABC \cdots XY) + \overline{AB}N(CD \cdots XY) \\ &\quad + \overline{AB}N(BD \cdots XY) + \cdots \\ &\quad + \overline{AY}N(BC \cdots X) , \end{aligned} \tag{B7}$$

and we wish to show that

$$\begin{aligned} AN(BC \cdots XYZ) &= N(ABC \cdots XYZ) \\ &\quad + \overline{AB}N(CD \cdots XYZ) \\ &\quad + \overline{AC}N(BD \cdots YZ) \\ &\quad + \overline{AZ}N(BC \cdots XY) . \end{aligned} \tag{B8}$$

The proof was as follows: First we take $Z = b$, and consider the left-hand side of (B8):

$$\begin{aligned} AN(BC \cdots XYb) &= A[N(BC \cdots XY)ca \\ &\quad - s\tilde{a}^\dagger N(BC \cdots XY)] . \end{aligned} \tag{B9}$$

This is obtained by simply inserting the N ordering of the operators.

Note that if $A = b$, then

$$A\tilde{a}^\dagger = \tilde{a}^\dagger A , \tag{B10}$$

while if $A = b^\dagger$, then

$$A\tilde{a}^\dagger = \tilde{a}^\dagger A - s . \tag{B11}$$

Thus we have for $Z = b$ and A either b or $b^\dagger$,

$$-sA\tilde{a}^\dagger = s\tilde{a}^\dagger A + \overline{AZ} . \tag{B12}$$

Returning to (B9), we have, for $Z = b$ and using (B8)

$$\begin{aligned} AN(BC \cdots XYZ) &= [N(AB \cdots XY) + \overline{AB}N(C \cdots XY) + \overline{AC}N(BD \cdots XY) \cdots + \overline{AY}N(BC \cdots X)]ca \\ &\quad - s\tilde{a}^\dagger[N(AB \cdots XY) + \overline{AB}N(C \cdots XY) \cdots \overline{AY}N(C \cdots X)] + \overline{AZ}N(B \cdots XY) . \end{aligned} \tag{B13}$$

We now reabsorb ca and $-s\tilde{a}^\dagger$ in the N product to get

$$AN(BC \cdots XYZ) = N(AB \cdots YZ) + \cdots - \overline{AB}N(C \cdots Z) + \overline{AZ}N(BC \cdots Y) . \tag{B14}$$

A similar proof goes through for $Z = b^\dagger$. Now the case of $N = 2$ was shown in (a) above. Q.E.D.

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